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# A limit theorem for tagged particles in a class of self-organizing particle systems

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The dynamics of tagged particles in a class of models which can exhibit nontrivial scaling behavior (self-organized criticality (SOC)) is investigated. Previously it was shown that in the hydrodynamic limit these models are described by diffusion equations with singular diffusion coefficients—a fact which explains the self-organizing behavior. Here we develop an alternate means for identifying SOC in these systems. We establish a functional central limit theorem for the rescaled position of a tagged particle in each model, and we establish asymptotics for the variance of the limiting Brownian motion as the density approaches unit (critical) density. We expect these methods will provide a useful means of characterizing the dynamics in related self-organizing systems.

tagged particles * self-organized criticality * functional central limit theorem

## 1. Introduction

Recently there has been a lot of interest in a class of particle systems which dynamically generate nontrivial scaling behavior. The prototypical examples are the so-called sand pile automata, introduced by Bak, Tang and Wiesenfeld (1987). In these systems 'sand' is added slowly to a pile, and when the local slope exceeds a threshold, sand falls according to a prescribed set of rules. Numerically they obtained a power law distributions of avalanche sizes, reminiscent of traditional equilibrium systems at a critical point. They termed this behavior 'self-organized criticality' (SOC).

In an effort to better understand the nature of the criticality in these and related systems, Carlson, Grannan, Swindle, and Tour (1990) introduced and studied a

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class of long range particle systems—two-state models—which exhibit SOC. These systems have the advantage that they are mathematically tractable, since they are both reversible and attractive. Furthermore, many of the results that can be obtained analytically for the two-state models are seen numerically to apply to sand pile models as well (see Carlson, Chayes, Grannan and Swindle, 1990). Careful analysis of the behavior of the two-state models leads to tests that can be used to characterize other self-organizing systems. Previously it was shown that the hydrodynamic limit of these self-organizing models are described by diffusion equations, in which the diffusion coefficients have singularities at a critical density. In this paper we analyze the dynamics of tagged particles, and show that under diffusion rescaling the paths of the tagged particles converge to Brownian motions in which the variance can diverge at the critical (unit) density. We expect that analysis of tagged particles may be particularly useful in distinguishing between systems in which power law scaling can be associated with self-organization and those which exhibit a generic scale invariance associated with more traditional (nonsingular) conservative diffusive systems (see Hwa and Kardar, 1989; Garrido, Lebowitz, Maes and Spohn, 1990; and Grinstein, Lee and Sachdev, 1990). Our primary motivation for considering limit laws for tagged particles is that the motion of a tagged particle can be considered in perturbations of these systems in which the jump or avalanche does not occur instantaneously (Carlson, Grannan and Swindle, 1991). In such systems the notion of a discrete event is no longer meaningful.

The two-state models are defined as follows: At time $t$ each site $x$ on the one-dimensional integer lattice $Z$ is either occupied ($\zeta_t(x)=1$) or vacant ($\zeta_t(x)=0$), where the configuration at time $t$ is $\zeta_t \in \{0, 1\}^Z$. The two-state models are distinguished from one another by their respective jump rules. In a given system, each 1 hops to the nearest vacant site in the positive direction at rate $c(k)$ if the target 0 is at distance $k$. The symmetric rule holds in the negative direction. In each model $c(k)$ is a nonnegative, nonincreasing function which defines the process.

Hydrodynamic limits for these models were obtained in Carlson, Grannan, Swindle and Tour (1990), where it was shown that under the usual diffusion scaling $(\varepsilon x, \varepsilon^{-2}t)$, the empirical measure, in which mass $\varepsilon$ is associated with each occupied site, converges weakly to the unique weak solution of

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left[D(\rho)\frac{\partial \rho}{\partial x}\right] \tag{1}$$

where

$$D(\rho) = \sum_{k=1}^{\infty} k^2 c(k) \rho^{k-1} . \tag{2}$$

Interestingly, if the function $c(k)$ decays sufficiently slowly, i.e. if

$$c(k) \sim 1/k^{\alpha} \quad \text{where} \quad 0 \leqslant \alpha \leqslant 3,$$

it follows from (2) that $D(\rho)$ has a singularity at the critical density $\rho_c = 1$. The order $\phi$ of the singularity depends on $\alpha$: $\phi = 3 - \alpha$ with a logarithmic singularity when $\alpha = 3$. In Carlson, Chayes, Grannan and Swindle (1990) it was shown that this diffusion singularity explains the observed SOC on open driven systems, and that the scaling behaviors of various observables are simply related to $\phi$. Of particular interest is the distribution of jump sizes, which is most directly related to the distribution of avalanche sizes in the sand pile models. It was shown that steady state solutions of (1) with appropriate boundary conditions lead to an asymptotic expression for $P_N(k)$, the probability of a jump of size $k$ in a system of size $N$ as $N \to \infty$, which exhibits finite-size scaling. The finite size scaling exponents were obtained in terms of $\phi$. It was suggested that singular diffusions, and, in particular, the order of the diffusion pole, may be one possible means by which to classify self-organizing systems.

In the study of interacting particle systems with a conservation law, limit laws for tagged particles have been a subject of interest for many years. For symmetric systems the goal is to discern the correct rescaling to obtain a limiting nondegenerate diffusion. So far most of the attention has been focused on the exclusion process, for which the rescaling can be nontrivial due to the impedance of the motion of a tagged particle by neighboring particles. It was shown in Arratia (1983) that for the symmetric nearest neighbor exclusion process in one dimension, the proper rescaling of the tagged particle position $x_t$ is $\varepsilon x_{t\varepsilon^{-4}}$. Subsequently in Kipnis and Varadhan (1986) it was shown that in all other cases, namely higher dimensions or non-nearest-neighbor interactions, the usual diffusion rescaling $\varepsilon x_{t\varepsilon^{-2}}$ results in convergence to a nondegenerate Brownian motion. For the two-state models that we are considering, it is this second scaling (the usual diffusion scaling) that is appropriate, and the proof which verifies that the limiting process obtained through this rescaling is nondegenerate is straightforward. Our main task in this paper is to extend this result to obtain the correct asymptotics for the variance of the limiting Brownian motion as the density approaches unity. In the limit of instantaneous transitions, scaling of the variance is analogous to scaling behavior which can be deduced from $P_N(k)$. However, unlike $P_N(k)$, motion of a tagged particle can be generalized to nonzero relaxation times.

In this paper we consider the class of two-state models described above (in the limit of instantaneous relaxation times). The formal generator of the process can be written for cylinder functions $f$ (i.e. functions which depend upon the configuration at a finite number of sites):

$$Lf(\eta) = \sum_{u<v} [f(\eta^{u,v}) - f(\eta)]\Pi(u, v)c(v-u) \tag{3}$$

where $\eta^{u,v}$ corresponds to the configuration $\eta$ with the states at sites $u$ and $v$ switched,

$$\eta^{u,v}(x) = \begin{cases} \eta(x) & \text{if } x \neq u, v, \\ \eta(v) & \text{if } x = u, \\ \eta(u) & \text{if } x = v, \end{cases} \tag{4}$$

and where

$$\Pi(u, v) = \prod_{u<k<v} \eta(k) \tag{5}$$

takes the value $\Pi(u, v) = 1$ when all sites between $u$ and $v$ are occupied. The fact that $Lf$ defines a process follows from methods in Liggett (1980), which rely heavily on the fact that $\eta_t$ is attractive when $c(k)$ is nonincreasing (see Lemma 3.2 in Carlson, Grannan, Swindle and Tour, 1990). A brief summary of the construction issues are discussed in the appendix. In addition to being attractive, the processes $\eta_t$ are reversible with respect to product measure $\nu_\rho$ for all $\rho < 1$.

In this paper we prove that the motion of a tagged particle starting with the system in equilibrium $\nu_\rho$ on the one-dimensional integer lattice $Z$ converges to a Brownian motion. We will take, for convenience, $c(k) = 1/k^\alpha$ where $\alpha \geq 0$, and we will establish asymptotics of the variance of the limiting diffusion as $\rho \to 1$—if the decay of the jump rate $c(k)$ is slow enough ($\alpha < 2$) then the variance will exhibit a singularity. Denoting the position of tagged particle by $x_t$, we prove:

**Theorem 1.1.** *Consider the process described by* (3) *with* $c(k) = k^{-\alpha}$, *and* $\alpha \geq 0$. *Let the initial distribution of the system be product measure* $\nu_\rho$ *at density* $\rho < 1$. *The rescaled tagged particle position* $\varepsilon x_{\varepsilon^{-2}t}$ *converges in distribution to a nondegenerate Brownian motion* $B_t$ *as* $\varepsilon \to 0$ *with variance* $\sigma^2(\rho)$, *where*

$$\sigma^2(\rho) \sim (1-\rho)^{-(2-\alpha)} \quad \text{as } \rho \to 1. \tag{6}$$

**Remarks.** (i) The equation $f \sim (1-\rho)^\beta$ means that $(1-\rho)^{-\beta} f(\rho) \to \kappa$ as $\rho \to 1$, where $\kappa$ is a strictly positive constant.

(ii) The asymptotics in (6) can be deduced from a simple nonrigorous calculation. In the invariant measure $\nu_\rho$, the variance in the size of the *first* jump of the tagged particle is: $2\sum_{k=1}^{\infty} k^2 c(k) \rho^{k-1}(1-\rho)$, which yields the same asymptotics given in (6). In fact, the constant prefactor ($\kappa$ above) obtained from this expression is also correct. The reason that this is the case, is that in the proof that follows, the position of the tagged particle is split into two pieces. The variance of one of the terms is easy to calculate and equals the sum above. Most of the work of the proof is then showing that the remaining piece has divergences at most of lower order.

## 2. Proof of Theorem 1.1

We begin with a brief sketch of the proof of Theorem 1.1. There are two parts to the proof: (i) to show that the process described by (3) converges to a Brownian motion with the usual rescaling, and (ii) to obtain bounds on the variance. Denote the position of the tagged particle at time $t$ by $x_t$. If $x_t$ were a martingale, a functional central limit theorem (part (i) of the proof) would follow immediately from results in Helland (1982). Unfortunately, $x_t$ is not a martingale. The goal, however, is to show that $x_t$ is the sum of a martingale and an inconsequential error term. Once this is done, the functional central limit theorem is immediate, and the only remaining task is to put bounds on the variance (part (ii) of the proof).

To accomplish the decomposition of the process into a martingale and error term, we need the following result in Kipnis and Varadhan (1986) which was established to analyze the tagged particle for the exclusion process. In reading Theorem 2.1, one should picture the Markov process $y(t)$ as corresponding to the particle system around the tagged particle, and $V(y(t))$ as corresponding to the expected jump distance of the tagged particle in the configuration at time $t$. For notational convenience we will at times use the notation $\langle f, g\rangle = \int fg \, d\nu_p$. Additionally, when a function $\phi$ is in the domain of $(-L)^{-1/2}$ we will write $\langle (-L)^{-1/2}\phi, (-L)^{-1/2}\phi\rangle = \langle \phi, (-L)^{-1}\phi\rangle$.

**Theorem 2.1** (Theorem 1.8 in Kipnis and Vardhan, 1986). *Let $y(t)$ be a Markov process, reversible and ergodic with respect to measure $\pi$. Let $V$ be a function in $L^2(X, \pi)$ such that both $\int V\pi = 0$ and $V \in \mathrm{Ran}(-L^{1/2})$, where $L$ is the generator of $y$ (this is equivalent to $|\langle V, F\rangle| \leq C\langle -LF, F\rangle^{1/2}$ for all $F$ in the domain of $L$). Consider*

$$X(t) = \int_0^t V(y(s)) \, ds. \tag{7}$$

*Then there exists a martingale $M(t)$ with respect of the usual filtration such that*

$$\lim_{t\to\infty} \frac{1}{t^{1/2}} \sup_{0\leq s\leq t} |X(s) - M(s)| = 0 \tag{8}$$

*and*

$$\lim_{t\to\infty} \frac{1}{t} E^P |X(t) - M(t)|^2 = 0 \tag{9}$$

*where $E^P$ denotes expectation with respect to the measure $P$ for the stationary Markov process with marginal $\pi$.* □

An immediate consequence is:

**Corollary 2.2.** *With the conditions of Theorem 2.1, under the usual diffusion scaling $\varepsilon^{1/2}X(\varepsilon^{-1}t)$ converges to a Brownian motion with variance $\sigma^2 = 2\langle -L^{-1}V, V\rangle$ as $\varepsilon \to 0$.* □

Before proceeding, we introduce the following notation. Recall that the configuration of the process at time $t$ is denoted by $\zeta_t$, and the location of the tagged particle at this time is $x_t$. Alternatively, we may define the configuration in the reference frame of the tagged particle by

$$\eta_t(u) = \zeta_t(x_t + u), \tag{10}$$

where $u$ denotes the position of each site relative to the tagged particle. In other words, in the $\eta_t$ process, the tagged particle is always at the origin. We will now concentrate on the process $(x_t, \eta_t)$ where $\eta_t \in \{0, 1\}^{Z-\{0\}}$ ($Z-\{0\}$ is the integer lattice without the origin) with generator

$$\tilde{L}f(x, \eta) = \sum_{z \neq 0} c(z)[1-\eta(z)]\Pi(0, z)[f(x+z, \tau_{-z}\eta) - f(x, \eta)] + \sum_{\substack{u<v \\ u,v\neq 0}} c(|u-v|)\Pi(u, v)[f(x, \eta^{u,v}) - f(x, \eta)], \tag{11}$$

where $\tau_{-z}$ shifts the origin to the site where the tagged particle has jumped:

$$\tau_{-z}\eta(u) = \begin{cases} \eta(u+z) & \text{if } u \neq -z, \\ 0 & \text{if } u = -z. \end{cases} \tag{12}$$

The first term in (11) corresponds to jumps of the tagged particle; the second term corresponds to jumps of particles other than the tagged particle. If we take $f(x, \eta) = f(\eta)$ (i.e. $f$ depends only on the configuration, and not on the location of the tagged particle) then we see that $\eta_t$ is a Markov process with generator

$$\begin{aligned} L_0 f(\eta) &= \sum_{z\neq 0} c(|z|)[1-\eta(z)]\Pi(0, z)[f(\tau_{-z}\eta) - f(\eta)] \\ &\quad + \sum_{\substack{u<v \\ u,v\neq 0}} c(|u-v|)\Pi(u, v)[f(\eta^{u,v}) - f(\eta)] \\ &\equiv L_1 f + L_2 f. \end{aligned} \tag{13}$$

The following lemma establishes the reversibility and ergodicity of the $\eta_t$ process:

**Lemma 2.3.** *The process $\eta_t$ defined in* (13) *is reversible and ergodic with respect to product measure $\nu_\rho$.*

**Proof.** We need to establish that $\int fL_0 g \, d\nu_\rho = \int gL_0 f \, d\nu_\rho$, for all cylinder functions $f$ and $g$, and we will check this individually for $L_1$ and $L_2$ (see (13)). The result for $L_2$ follows from the fact that

$$\int f(\eta)\Pi(u, v)g(\eta^{u,v}) \, d\nu_\rho = \int g(\eta)\Pi(u, v)f(\eta^{u,v}) \, d\nu_\rho. \tag{14}$$

The result for $L_1$ follows from

$$\begin{aligned} &\int f(\eta)c(|z|)[1-\eta(z)]\Pi(0, z)g(\tau_{-z}\eta) \, d\nu_\rho \\ &= \int g(\eta)c(|-z|)[1-\eta(-z)]\Pi(0, -z)f(\tau_z\eta) \, d\nu_\rho \end{aligned} \tag{15}$$

which is a consequence of the fact that the mapping: $\eta \to \tau_z\eta$ takes $[1-\eta(z)]\Pi(0, z)$ to $[1-\eta(-z)]\Pi(0, -z)$. Ergodicity follows exactly as in Kipnis and Varadhan (1986). $\square$

We will now decompose $x_t$ into a martingale and a correction. We know that

$$N_t = x_t - \int_0^t \phi(\eta_s) \, ds \tag{16}$$

is a martingale, where $\phi$ is the generator $\tilde{L}$ given in (11) applied to the location of the tagged particle,

$$\phi(\eta) \equiv \tilde{L}(x) = \sum_{z \neq 0} zc(|z|)[1 - \eta(z)]\Pi(0, z). \tag{17}$$

Following Kipnis and Varadhan (1986), our goal is to use Theorem 2.1 to write

$$\int_0^t \phi(\eta_s)\,\mathrm{d}s = M_t + E_t \tag{18}$$

where $M_t$ is a martingale and $E_t$ is a vanishing error, which implies the desired decomposition: $x_t = M_t + N_t + E_t$. In order to use Theorem 2.1, we must establish the following Lemma 2.4 which shows that $\phi \in \mathrm{Ran}((-L)^{1/2})$. This lemma will also be used to put lower bounds on the variance of the limiting diffusion, thereby establishing nondegeneracy. Furthermore, Lemma 2.4 contains the asymptotics for a prefactor as $\rho \to 1$, which is used to establish the correct asymptotics of the variance for the diffusion.

**Lemma 2.4.** *Take $c(k) = 1/k^\alpha$ for fixed $\alpha \geq 0$, and let the density $\rho < 1$. There exists a constant $A$ so that, for any $F$ in the domain of $L$, the following bound holds:*

$$\begin{aligned}\left|\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\right| &\leq \frac{A}{(1-\rho)^{1/2-\alpha/2}}\left[-\int F(\eta)L_2F(\eta)\,\mathrm{d}\nu_\rho\right]^{1/2}\\ &\leq \frac{A}{(1-\rho)^{1/2-\alpha/2}}\left[-\int F(\eta)L_0F(\eta)\,\mathrm{d}\nu_\rho\right]^{1/2}\end{aligned} \tag{19}$$

*where $L_0$ and $L_2$ are defined in* (13).

*An instructive calculation*: We begin with a simple calculation which motivates the actual proof. It is in fact very easy to show that

$$\left|\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\right| \leq \frac{A}{(1-\rho)^{1-\alpha/2}}\left[-\int F(\eta)L_0F(\eta)\,\mathrm{d}\nu_\rho\right]^{1/2} \tag{20}$$

(note the change in the exponent of the prefactor in the right hand side), and it is, in fact, instructive to do this. Taking $F(\eta)$ a cylinder function, by definition,

$$\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho = \int \sum_{z\neq 0} zc(|z|)[1-\eta(z)]\Pi(0,z)F(\eta)\,\mathrm{d}\nu_\rho. \tag{21}$$

Symmetrizing we have

$$\begin{aligned}&\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\\ &= \int \sum_{z=1}^{\infty} zc(z)\{[1-\eta(z)]\Pi(0,z) - [1-\eta(-z)]\Pi(0,-z)\}F(\eta)\,\mathrm{d}\nu_\rho\\ &= \int \sum_{z=1}^{\infty} zc(z)[1-\eta(z)]\Pi(0,z)[F(\eta) - F(\tau_{-z}\eta)]\,\mathrm{d}\nu_\rho\end{aligned} \tag{22}$$

where the second equality follows from the fact that $\tau_z\eta$ maps $[1-\eta(z)]\Pi(0,z)\to[1-\eta(-z)]\Pi(0,-z)$. Using Holder's inequality we obtain the following bound:

$$\begin{aligned}&\left|\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\right|\\&\quad\leqslant\left\{\int\sum_{z=1}^{\infty} z^2c(z)[1-\eta(z)]\Pi(0,z)\,\mathrm{d}\nu_\rho\right\}^{1/2}\\&\qquad\times\left\{\int\sum_{z=1}^{\infty} c(z)[1-\eta(z)]\Pi(0,z)[F(\eta)-F(\tau_{-z}\eta)]^2\,\mathrm{d}\nu_\rho\right\}^{1/2}.\end{aligned}\tag{23}$$

The first term is a constant: $A=\{(1-\rho)\sum_{k=1}^{\infty}k^2c(k)\rho^{k-1}\}^{1/2}<\infty$ since $\rho<1$. Using reversibility, we have

$$\left|\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\right|\leqslant A\left\{\int F(-L_1F)\,\mathrm{d}\nu_\rho\right\}^{1/2}.\tag{24}$$

This simple computation is sufficient to allow use of Theorem 2.1. However, not only is this result not enough to establish nondegeneracy of the limiting diffusion, but stopping here would also preclude establishing the correct asymptotics of the variance as $\rho\to1$, since this weaker form of the estimate would result in the limiting variance of $\int_0^t\phi(\eta_s)\,\mathrm{d}s$ which has the same order singularity as $N_t$, which means that we could not exclude the possibility that the leading singularities of two pieces cancel.

**Proof of Lemma 2.4.** Let $z_l(\eta)$ and $z_r(\eta)$ denote the position of the nearest vacant site to the left and to the right of the origin (tagged particle) in configuration $\eta$. We decompose the state space into sets of configuration according to the relative distances of the vacant sites from the origin:

$$\mathcal{N}=\{\eta:|z_l|<|z_r|\},\qquad \mathcal{P}=\{\eta:|z_l|>|z_r|\},\qquad \mathcal{S}=\{\eta:|z_l|=|z_r|\}.\tag{25}$$

Referring to (22), we begin with

$$\begin{aligned}&\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\\&\quad=\sum_{z=1}^{\infty} zc(|z|)\int\{[1-\eta(z)]\Pi(0,z)-[1-\eta(-z)]\Pi(0,-z)\}F(\eta)\,\mathrm{d}\nu_\rho\\&\quad=\sum_{z=1}^{\infty} zc(|z|)\left\{\int_{\mathcal{P}}[1-\eta(z)]\Pi(0,z)F(\eta)\,\mathrm{d}\nu_\rho\right.\\&\qquad\qquad-\int_{\mathcal{N}}[1-\eta(-z)]\Pi(0,-z)F(\eta)\,\mathrm{d}\nu_\rho\\&\qquad\qquad+\int_{\mathcal{N}}[1-\eta(z)]\Pi(0,z)F(\eta)\,\mathrm{d}\nu_\rho\\&\qquad\qquad\left.-\int_{\mathcal{P}}[1-\eta(-z)]\Pi(0,-z)F(\eta)\,\mathrm{d}\nu_\rho\right\}.\end{aligned}\tag{26}$$

Terms involving $S$ cancel identically and are omitted. We will continue to work with the first two terms in the sum in the last expression—an identical argument dispenses with the remaining two terms. These terms can be written as

$$\sum_{z=1}^{\infty} zc(|z|) \int \{\eta(-z)\Pi(-z,z)[1-\eta(z)] \\ -[1-\eta(-z)]\Pi(-z,z)\eta(z)\}F(\eta)\,\mathrm{d}\nu_\rho \tag{27}$$

where the integral once again ranges over the entire state space, since, for example, $\sum_{z=1}^{\infty} \eta(-z)\Pi(-z,z)[1-\eta(z)]$ automatically restricts the region of integration to $\mathscr{P}$ in the first term. The next step is to rewrite the above expression as a sum of telescoping sequences. Each term corresponds to a bijective mapping of $\mathscr{N}$ to $\mathscr{P}$, and is defined by two consecutive switchings of the states at pairs of sites, which, when applied to a configuration in $\mathscr{N}$ yields a configuration in $\mathscr{P}$. More precisely, given a configuration in $\mathscr{N}\cap\{|z_t|=z\}$, for any $i$ with $1\leqslant i\leqslant z-1$, the mapping, which we will call $M_i^{(z)}$, is defined by

$$M_i^{(z)}(\eta)=[\eta^{(-z,i)}]^{(i,z)} \tag{28}$$

in which the 1 at site $i$ is moved to the vacant site $-z$, and then the 1 at $z$ is moved to site $i$. With this in mind, (27) can be written as

$$\begin{aligned}
&\sum_{z=1}^{\infty} zc(|z|) \int \frac{1}{z-1}\sum_{i=1}^{z-1} \{\eta(-z)\Pi(-z,z)[1-\eta(z)] \\
&\qquad -\eta(-z)\Pi(-z,i)[1-\eta(i)]\Pi(i,z)\eta(z) \\
&\qquad +\eta(-z)\Pi(-z,i)[1-\eta(i)]\Pi(i,z)\eta(z) \\
&\qquad -[1-\eta(-z)]\Pi(-z,z)\eta(z)\}F(\eta)\,\mathrm{d}\nu_\rho \\
&=\sum_{z=1}^{\infty} zc(|z|) \int \frac{1}{z-1}\sum_{i=1}^{z-1} \{\eta(-z)\Pi(-z,z)[1-\eta(z)][F(\eta)-F(\eta^{(i,z)})] \\
&\qquad +[1-\eta(-z)]\Pi(-z,z)\eta(z) \\
&\qquad \times[F(\eta^{(-z,i)})-F(\eta)]\}\,\mathrm{d}\nu_\rho
\end{aligned} \tag{29}$$

where the last equality follows from the following fact:

$$\begin{aligned}
&\int \{\eta(i)\Pi(i,i+j)[1-\eta(i+j)]-[1-\eta(i)]\Pi(i,i+j)\eta(i+j)\}F(\eta)\,\mathrm{d}\nu_\rho \\
&=\int [1-\eta(i)]\Pi(i,i+j)\eta(i+j)[F(\eta^{i,i+j})-F(\eta)]\,\mathrm{d}\nu_\rho.
\end{aligned} \tag{30}$$

This is a consequence of a change of variables $\eta\to\eta^{i,i+j}$ in the first term of the integrand in (30), and the fact that the integral is with respect to product measure $\nu_\rho$.

To simplify notation, we will index the first term in the integrand in (29) by $J=1$ denoting the first jump, and the second term by $J=2$ denoting the second jump. The change in $F$ will be denoted by $\Delta F_J$, and the configurational prefactor will be denoted by $\Pi_J^{(z)}$. We will also denote the transition rate for jump $J$ by $c_J$ (e.g. $c_1 = c(z-i)$). We now have

$$\begin{aligned}\left|\int \phi(\eta)F(\eta)\,\mathrm{d}\nu_\rho\right| &= \left|\sum_{z=1}^{\infty}\int\sum_{i=1}^{z-1}\sum_{J=1,2}\left(\frac{zc(z)}{z-1}\right)\Pi_J^{(z)}\Delta F_J\,\mathrm{d}\nu_\rho\right| \\ &\leqslant \left\{\sum_{z=1}^{\infty}\int\sum_{i=1}^{z-1}\sum_{J=1,2}\left(\frac{zc(z)}{z-1}\right)^2\frac{\Pi_J^{(z)}}{c_J}\,\mathrm{d}\nu_\rho\right\}^{1/2} \\ &\quad\times\left\{\sum_{z=1}^{\infty}\int\sum_{i=1}^{z-1}\sum_{J=1,2}\Pi_J^{(z)}c_J\Delta F_J^2\,\mathrm{d}\nu_\rho\right\}^{1/2} \end{aligned} \tag{31}$$

where the last step follows from Holder's inequality. The last factor in (31) is clearly bounded by

$$\left(\int F(-L_2F)\,\mathrm{d}\nu_\rho\right)^{1/2} = \left(\int\sum_{u<v} c(v-u)\Pi(u,v)\{F(\eta^{u,v})-F(\eta)\}^2\,\mathrm{d}\nu_\rho\right)^{1/2}. \tag{32}$$

All that remains is to bound

$$\begin{aligned}&\left\{\sum_{z=1}^{\infty}\int\sum_{i=1}^{z-1}\sum_{J=1,2}\left(\frac{zc(z)}{z-1}\right)^2\frac{\Pi_J^{(z)}}{c_J}\,\mathrm{d}\nu_\rho\right\}^{1/2} \\ &\quad\leqslant\left\{\sum_{z=1}^{\infty}\int\sum_{i=1}^{z-1}\sum_{J=1,2}\left(\frac{zc(z)}{z-1}\right)^2\frac{1}{c(2z)}\Pi_J^{(z)}\,\mathrm{d}\nu_\rho\right\}^{1/2} \\ &\quad=\left\{\sum_{z=1}^{\infty}\left(\frac{z^2c(z)^2}{(z-1)^2c(2z)}\right)2(z-1)\rho^{2z-1}(1-\rho)\right\}^{1/2} \\ &\quad\leqslant\left\{8(2^\alpha)\sum_{z=1}^{\infty}z^{1-\alpha}\rho^{z-1}(1-\rho)\right\}^{1/2} \\ &\quad\leqslant A(1-\rho)^{-(1-\alpha)/2}\end{aligned} \tag{33}$$

where the first step involved replacing $c_J$ with the smaller $c(2z)$. This completes the proof of the lemma. □

**Remark on Proof of Lemma 2.4.** In the proof of Lemma 2.4, the fact that the number of paths (mappings) from $\mathcal{N}$ to $\mathcal{P}$ increased linearly with $z$ was essential. Using only one path results in an estimate of the prefactor that has a higher order singularity. Even though the result above is adequate for our purposes, one might be tempted to try to use mappings involving more than two jumps. However, this strategy does not lead to a better bound due to the appearance of repetitions in the $\Delta F_J^2$ terms which alter the bound by $\langle F, -L_2F\rangle^{1/2}$ in (31) and (32).

We complete the proof of the therorem in two parts:

*Asymptotics for the limiting variance*: We first bound the limiting variance of $\int_0^t \phi(\eta_s)\,\mathrm{d}s$ by using Lemma 2.4 to obtain

$$\langle \phi, (-L_0)^{-1}\phi\rangle \leqslant A^2(1-\rho)^{-(1-\alpha)} \tag{34}$$

which, by Corollary 2.2, bounds the limiting variance. To see (34) formally, set $F=(-L_0)^{-1}\phi$ in the bound (equation (19)):

$$|\langle \phi, F\rangle| \leqslant A(1-\rho)^{-(1-\alpha)/2}\langle F, -L_0F\rangle^{1/2}. \tag{35}$$

We next turn to the remaining component of the limiting diffusion corresponding to the term $N_t$ given in (16). There is a family of basic, mutually orthogonal martingales,

$$\mu_t^{u,v} = \sum_{s\leqslant t} 1_{\{\eta_{s-}=\eta_s^{u,v}\}}|\eta_s(u)-\eta_s(v)| - \int_0^t |\eta_s(u)-\eta_s(v)|c(|u-v|)\Pi(u,v)\,\mathrm{d}s \tag{36}$$

and

$$\mu_t^z = \sum_{s\leqslant t} 1_{\{\eta_s=\tau_{-z}\eta_{s-}\}} - \int_0^t c(|z|)\Pi(0,z)\,\mathrm{d}s. \tag{37}$$

The sums above simply count the number of jumps of the appropriate type that have occurred by time $t$. Noting that the position of the tagged particle $x_t$ is the sum over the possible jump distances of the product of the jump distance with number of such jumps that have occurred, and referring to (16), we can write

$$N_t = \sum_{z\neq 0} z\mu_t^z. \tag{38}$$

Using orthogonality, we see that

$$\operatorname{Var}(N_t) = E^{\nu_\rho}\left\{\sum_{z\neq 0} z^2(\mu_t^z)^2\right\} = E^{\nu_\rho}\left\{\sum_{z\neq 0} z^2 V(\mu_t^z)\right\} \tag{39}$$

where $V(\cdot)$ denotes the quadratic variation of the martingale. A simple calculation yields

$$V(\mu_t^z) = \int_0^t c(|z|)\Pi(0,z)[1-\eta(z)] \tag{40}$$

which immediately yields

$$\operatorname{Var}(N_t) = 2t\sum_{z=1}^{\infty} z^2 c(z)\rho^{z-1}(1-\rho) \sim (1-\rho)^{-(2-\alpha)}. \tag{41}$$

Equations (34) and (41) imply that the variance of the limiting diffusion $B(t)$ is asymptotically $(1-\rho)^{-(2-\alpha)}$ as desired.

*Nondegeneracy of the limiting diffusion for all densities*: We proceed to establish nondegeneracy following Kipnis and Varadhan (1986). The result will be a uniform lower bound on the variance for all densities $0 \leq \rho \leq 1$.

The goal is to show that $\int_0^t \phi(\eta_s)\,\mathrm{d}s = M_t + E_t$ has components of the $\mu^{u,v}$ martingales. Since $N_t$ is comprised solely of the family of $\mu^z$ martingales, and since all of these basic martingales are orthogonal, the limiting diffusion would then have to be nondegenerate, since the $N_t$ could not cancel $M_t$.

Recall the family of orthogonal martingales given in (36) and (37). For any function $F$ in the domain of $L_0$,

$$M_t^F = F(\eta(t)) - F(\eta(0)) - \int_0^t (L_0 F)(\eta(s))\,\mathrm{d}s \tag{42}$$

is a martingale, which can be expressed in terms of the orthogonal family as

$$\begin{aligned} M_t^F = {} & \sum_{u,v\neq 0} \int_0^t [F(\eta^{u,v}(s)) - F(\eta(s))]\,\mathrm{d}\mu_s^{u,v} \\ & + \sum_{z\neq 0} \int_0^t [F(\tau_{-z}\eta(s)) - F(\eta(s))]\,\mathrm{d}\mu_s^z . \end{aligned} \tag{43}$$

If we knew that $\phi$ was in the domain of $L_0^{-1}$, then we would proceed by setting $F = (-L_0)^{-1}\phi$ in the previous expressions, thereby obtaining a decomposition of $\int_0^t \phi(\eta_s)\,\mathrm{d}s$ in terms of the basic martingales. Since we do not know this, we will take $F_\lambda = (\lambda I - L_0)^{-1}\phi$ (recall that the resolvent set is $(0,\infty)$). To show that the limiting diffusion in the theorem has a nondegenerate variance, we need to make sure that the $\mu^{u,v}$ term is actually there—by orthogonality the $\mu^z$ terms cannot cancel it. This will follow once we show that

$$\begin{aligned} & \lim_{\lambda\to 0} E^{\nu_\rho}\left\{ \sum_{u,v\neq 0} c(|u-v|)\Pi(u,v)[F_\lambda(\eta^{u,v}(s)) - F_\lambda(\eta(s))]^2 \right\} \\ & = \lim_{\lambda\to 0} \langle -L_2(\lambda I - L_0)^{-1}\phi, (\lambda I - L_0)^{-1}\phi\rangle \neq 0 . \end{aligned} \tag{44}$$

This result is sufficient to obtain nondegeneracy. Observe that Lemma 2.4 implies

$$\langle (-L_2)^{-1}\phi, \phi\rangle \leq \frac{A^2}{(1-\rho)^{1-\alpha}} . \tag{45}$$

Defining $C(\rho) = A(1-\rho)^{-(1-\alpha)/2}$, observe that,

$$\begin{aligned} & \langle \phi, (\lambda I - L_0)^{-1}\phi\rangle \\ & = \langle (-L_2)^{-1/2}\phi, (-L_2)^{1/2}(\lambda I - L_0)^{-1}\phi\rangle \\ & \leq \langle (-L_2)^{-1}\phi, \phi\rangle^{1/2}\langle (-L_2)(\lambda I - L_0)^{-1}\phi, (\lambda I - L_0)^{-1}\phi\rangle^{1/2} \\ & \leq C(\rho)\langle (-L_2)(\lambda I - L_0)^{-1}\phi, (\lambda I - L_0)^{-1}\phi\rangle^{1/2}, \end{aligned} \tag{46}$$

where we have used Holder's inequality. This is equivalent to

$$\lim_{\lambda \to 0} \langle (-L_2)(\lambda I - L_0)^{-1}\phi, (\lambda I - L_0)^{-1}\phi \rangle \geq \frac{\langle \phi, (-L_0)^{-1}\phi \rangle^2}{C(\rho)^2}. \tag{47}$$

Now note that:

$$\begin{aligned} \langle \phi, \phi \rangle &= \langle (-L_0)^{1/2}\phi, (-L_0)^{-1/2}\phi \rangle \\ &\leq \langle (-L_0)^{1/2}\phi, (-L_0)^{1/2}\phi \rangle^{1/2} \langle (-L_0)^{-1/2}\phi, (-L_0)^{-1/2}\phi \rangle^{1/2} \\ &= \langle \phi, -L_0\phi \rangle^{1/2} \langle \phi, (-L_0)^{-1}\phi \rangle^{1/2} \end{aligned} \tag{48}$$

which amounts to

$$\langle \phi, (-L_0)^{-1}\phi \rangle \geq \frac{\langle \phi, \phi \rangle^2}{\langle \phi, -L_0\phi \rangle}. \tag{49}$$

Nondegeneracy is now immediate, since (49) implies that the right hand side of (47) is positive. □

**Remark.** With a little more work we can obtain a lower bound. An upper bound on the denominator in (47) is provided by Lemma 2.4. To put lower bounds on the numerator, recall (49). A simple computation shows that: $\langle \phi, \phi \rangle^2 \geq \gamma(1-\rho)^{-4(1-\alpha)}$, and a painful computation yields: $\langle \phi, -L_0\phi \rangle \leq \delta(1-\rho)^{-(3-2\alpha)}$. These facts, together with (47), (48) and (49) yield a lower bound for the variance of the limiting diffusion $B(t)$ which holds for all $\rho$:

$$\mathrm{Var}(B(t)) \geq \frac{\kappa}{(1-\rho)^{1-3\alpha}}. \tag{50}$$

## Appendix: Construction of the process

We give a brief summary of the issues involved in the construction of the processes that we consider in this paper. The reader can refer to Liggett (1980) for a detailed discussion of a class of long range exclusion processes which includes one of the systems discussed in this paper—namely the process with $c(k) = 1/k$. In what follows we can take $c(k)$ to be any nonnegative nonincreasing function—in particular, $c(k) = k^{-\alpha}$ for any $\alpha \geq 0$.

Given a configuration $\xi$, let $|\xi| = \sum_x \xi(x)$ denote the number of sites occupied by 1's in $\xi$. The process is well defined for any finite $\xi$ (i.e. $|\xi| < \infty$). Denote the semigroup by $S(t)f(\xi) = E^{\xi}f(\xi_t)$ for all $f \in C(X)$ were $X = \{0, 1\}^Z$. We will now outline the essential aspects of the construction of the semigroup for all configurations. We will note the places where arguments in Liggett (1980) must be altered for the processes that we consider.

(a) To define the semigroup on $\zeta$ with $|\zeta|=\infty$, take any increasing $f$ and define $S(t)f(\zeta)$ in terms of the limit of $S(t)f(\xi)$ with $\xi$ finite:

$$S(t)f(\zeta)=\lim_{\substack{\xi\uparrow\zeta\\|\xi|<\infty}} S(t)f(\xi). \tag{51}$$

(b) Let $Tf(\zeta)=\lim_{t\downarrow 0} S(t)f(\zeta)$. It is straightforward to show that this limit exists, that $Tf\geqslant f$ for increasing $f$, and that $Tf(\mathbf{1})=f(\mathbf{1})$ where $\mathbf{1}$ denotes the configuration in which all sites are occupied.

(c) The following set of configuration are those on which the process is reasonably well behaved:

$$D=\{\zeta\colon\ Tf(\zeta)=f(\zeta)\ \forall f\in C(X)\}. \tag{52}$$

One then characterizes $D$ by showing that if $\xi\leqslant\zeta$, $\zeta\in D$, and $\zeta\neq\mathbf{1}$, then $\xi\in D$. To do this requires the following result. The corresponding result in Liggett (1980) uses the random walk nature of the transition mechanism which is not present for general $\alpha$ in our systems.

**Lemma A.1.** *Let $\xi\leqslant\zeta$, $\xi(x)=0$, $\zeta(x)=1$, and $\xi(y)=0$, with $x<y$. Then*

$$\lim_{t\downarrow 0} P^{\zeta}(\zeta(y)=1)$$
$$\geqslant \mathrm{e}^{-t}\left\{\frac{1}{(1+|x-y|^{\alpha})}\right\}\Pi_{x<i<y}\zeta(i)\lim_{t\downarrow 0} P^{\xi}(\xi(y)=1). \qquad\square \tag{53}$$

(d) Next we mention two sequences of approximations to the process $S(t)$ which do not rely upon the random walk properties of the long range exclusion process. Given a positive integer $R$, consider the process in which a 1 can move no further than $R$ units away. Jump rates are unmodified, with the exception that at rate $c(R+1)$ each 1 attempts to jump outside of this range and is removed. Then generator is given by

$$\begin{aligned}\Omega_R f(\zeta)=\sum_{i\in Z}\Bigg\{&\sum_{j=-R}^{R}\zeta(i)\Pi(i,i+j)c(j)[f(\zeta^{i,i+j})-f(\zeta)]\\&+\zeta(i)\Pi(i,i+R+1)c(R+1)[f(\zeta^{i})-f(\zeta)]\\&+\zeta(i)\Pi(i,i-R-1)c(R+1)[f(\zeta^{i})-f(\zeta)]\Bigg\}\end{aligned} \tag{54}$$

where $\zeta^{i}$ denotes the configuration $\zeta$ with the spin at site $i$ flipped. The semigroup is denoted by $S_R(t)$. The second approximation denoted by $U_R(t)$ is simply the process in which jumps outside of the interval $[-R, R]$ are suppressed. It can be shown that:

**Lemma A.2.** *For all increasing $f$,*

$$S_R(t)f\leqslant S_{R+1}(t)f\leqslant S(t)f \tag{55}$$

*and, for all* $f \in C(X)$,

$$\lim_{R\to\infty} S_R(t)f = S(t)f. \qquad \square \tag{56}$$

(e) The approximations $U_R$ are used to show that: $\nu_\rho(D) = 1$. To see this, note that for finite $\xi$, $\lim_{R\to\infty} U_R(t)f(\xi) = S(t)f(\xi)$. Consequently, for any increasing $f$ and any $\zeta$,

$$S(t)f(\zeta) \leq \liminf_{R\to\infty} U_R(t)f(\zeta). \tag{57}$$

Observing that $\nu_\rho$ is invariant for the approximations $U_R$, we see that for increasing $f$,

$$\int S(t)f \, \mathrm{d}\nu_\rho \leq \int U_R(t)f \, \mathrm{d}\nu_\rho = \int f \, \mathrm{d}\nu_\rho \tag{58}$$

which implies that $\nu_\rho S(t) \leq \nu_\rho$. This monotonicity implies that $\int T(f) \, \mathrm{d}\nu_\rho \leq \int f \, \mathrm{d}\nu_\rho$. Noting that $Tf \geq f$ for increasing $f$ from (b) above, we have $\nu_\rho(D) = 1$, as desired.

(f) The final step is to show that $\Omega f = \lim_{R\to\infty} \Omega_R f$ is well defined, and to establish that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int S(t)\zeta(x) \, \mathrm{d}\nu_\rho = \int \Omega\zeta(x) \, \mathrm{d}[\nu_\rho S(t)] \tag{59}$$

which is then used to establish that $\nu_\rho$ is invariant for $S(t)$ for any constant $\rho < 1$. Invariance for $\rho = 1$ is a consequence of a simple monotonicity argument.

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